

# CLASSIFICATION OF EQUIVARIANT COMPLEX VECTOR BUNDLES OVER A CIRCLE

JIN-HWAN CHO, SUNG SOOK KIM, MIKIYA MASUDA, AND DONG YOUP SUH

**ABSTRACT.** In this paper we characterize the fiber representations of equivariant complex vector bundles over a circle and classify these bundles. We also treat the triviality of equivariant complex vector bundles over a circle by investigating the extensions of representations. As a corollary of our results, we calculate the reduced equivariant  $K$ -group of a circle for any compact Lie group.

## 1. INTRODUCTION

The classification of vector bundles over a topological space is one of the fundamental problems in topology, and many theories have been developed to solve the problem. However the problem becomes more complex and difficult when one considers it in the equivariant category. For instance, every complex vector bundle over a circle is trivial, but equivariant ones are abundant and not necessarily trivial. In this paper, we classify equivariant complex vector bundles over a circle. The real case is treated in another paper [CKMS99].

In order to state our main results, let us fix some notation and terminology. Let  $G$  be a compact Lie group and let  $\rho: G \rightarrow O(2)$  be an orthogonal representation of  $G$ . The unit circle of the corresponding  $G$ -module is denoted by  $S(\rho)$ . We set  $H = \rho^{-1}(1)$ , so that  $H$  acts trivially on  $S(\rho)$  and the fiber  $H$ -module of a complex  $G$ -vector bundle over  $S(\rho)$  is determined uniquely up to isomorphism. On the other hand, for a character  $\chi$  of  $H$  and  $g \in G$ , a new character  ${}^g\chi$  of  $H$  is defined by  ${}^g\chi(h) = \chi(g^{-1}hg)$  for  $h \in H$ . We say that the character  $\chi$  is  $G$ -invariant if  ${}^g\chi = \chi$  for all  $g \in G$ . Our first main theorem characterizes the fiber  $H$ -module of a complex  $G$ -vector bundle over  $S(\rho)$ .

**Theorem A.** *A complex  $H$ -module is the fiber  $H$ -module of a complex  $G$ -vector bundle over  $S(\rho)$  if and only if its character is  $G$ -invariant.*

We need more notation to state our second main theorem. Let  $\text{Irr}(H)$  be the set of characters of irreducible  $H$ -modules. It has a  $G$ -action defined above. Since a character is a class function, the isotropy subgroup  $G_\chi$  of  $G$  at  $\chi \in \text{Irr}(H)$  contains  $H$ . We

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choose and fix a representative from each  $G$ -orbit in  $\text{Irr}(H)$  and denote the set of those representatives by  $\text{Irr}(H)/G$ . Denote by  $\text{Vect}_G(X)$  the set of isomorphism classes of complex  $G$ -vector bundles over a connected  $G$ -space  $X$  and by  $\text{Vect}_{G_\chi}(X, \chi)$  the subset of  $\text{Vect}_{G_\chi}(X)$  with a multiple of  $\chi$  as the character of fiber  $H$ -modules. They are semi-groups under Whitney sum. The decomposition of a  $G$ -vector bundle into the  $\chi$ -isotypical components induces an isomorphism

$$\text{Vect}_G(X) \cong \prod_{\chi \in \text{Irr}(H)/G} \text{Vect}_{G_\chi}(X, \chi).$$

This reduces the study of  $\text{Vect}_G(X)$  to that of  $\text{Vect}_{G_\chi}(X, \chi)$  (see Section 2).

**Theorem B.** *The semi-group  $\text{Vect}_{G_\chi}(S(\rho), \chi)$  is generated by*

- (1) *one element  $L_\chi$  if  $\rho(G_\chi) \subset SO(2)$ ,*
- (2) *two elements  $L_\chi^\pm$  if  $\rho(G_\chi) = O(2)$ , and*
- (3) *four elements  $L_\chi^{\pm\pm}$  with a relation  $L_\chi^{++} + L_\chi^{--} = L_\chi^{+-} + L_\chi^{-+}$  otherwise.*

Using this theorem, one can easily enumerate complex  $G$ -vector bundles over  $S(\rho)$  with a fixed  $H$ -module as the fiber  $H$ -module (see Corollary 5.2).

Our last main theorem is about the triviality of  $G$ -vector bundles. Here a  $G$ -vector bundle over  $X$  is said to be *trivial* if it is isomorphic to a product bundle with a  $G$ -module as its fiber.

**Theorem C.** *The triviality of the generators appeared in Theorem B is as follows.*

- (1)  *$L_\chi$  is trivial.*
- (2)  *$L_\chi^\pm$  are both trivial or both nontrivial.*
- (3) *Two of  $L_\chi^{\pm\pm}$  are trivial and the other two are nontrivial if  $|\rho(G_\chi)|/2$  is odd, and  $L_\chi^{\pm\pm}$  are all trivial or all nontrivial if  $|\rho(G_\chi)|/2$  is even, where  $|\rho(G_\chi)|$  denotes the order of the dihedral group  $\rho(G_\chi)$ .*

Since  $\rho(G_\chi) \subset SO(2)$  for any  $\chi$  if  $\rho(G) \subset SO(2)$ , it follows that

**Corollary D.** *Every  $G$ -vector bundle over  $S(\rho)$  is trivial if  $\rho(G) \subset SO(2)$ .*

The reader will find that there are many nontrivial  $G$ -vector bundles as well as trivial ones unless  $\rho(G) \subset SO(2)$ .

This paper is organized as follows. Sections 2 and 3 deal with results on  $G$ -vector bundles which hold for an arbitrary base space. We also recall some results from representation theory, which turn out to be closely related to the semi-group structure on  $\text{Vect}_G(S(\rho))$  and the triviality of  $G$ -vector bundles over  $S(\rho)$ . Theorems A and B are proved in Sections 4 and 5. In Section 6 we present another approach to study  $\text{Vect}_G(S(\rho))$ . The triviality of  $G$ -vector bundles over  $S(\rho)$  is discussed and the proof of Theorem C is given in Section 7. We describe  $G$ -line bundles explicitly in Section 8. In Section 9 we apply the general results obtained in the previous sections to the case when  $G$  is abelian. In Section 10 we determine the reduced equivariant  $K$ -group of  $S(\rho)$ , which extends a result of Y. Yang [Yan95] for a finite cyclic group to any compact Lie group  $G$ .

The subject treated in this paper is classical and the reader may wonder why we were led to study this subject. In fact, we were concerned with what is called the manifold

realization problem. It asks whether a closed smooth  $G$ -manifold is equivariantly diffeomorphic to a non-singular real affine  $G$ -variety. This problem was originally considered in the non-equivariant category by J. Nash [Nas52], and affirmatively solved by A. Tognoli [Tog73]. Then R. Palais [Pal81] considered the equivariant case above, and some partial affirmative solutions are obtained, see [DM95] for instance. It is even considered to realize smooth  $G$ -vector bundles over closed smooth  $G$ -manifolds by algebraic ones, which is called the bundle realization problem, and some partial affirmative solutions are obtained as well, see [DMS94]. Apparently the latter problem is more general than the former, but they are linked. Namely we encounter the bundle realization problem to solve the manifold realization problem. For instance, we were faced with realizing real or complex  $G$ -line bundles over a circle by algebraic ones to solve the manifold realization problem for two- or three-dimensional manifolds [KM94, CS97]. This motivated us to investigate  $G$ -vector bundles over a circle. At the beginning of this research we suspected that the problem might already be solved, but there is no literature as far as we know. We hope that it is worth while publishing the results obtained in this paper in print.

## 2. DECOMPOSITION OF $G$ -VECTOR BUNDLES

Hereafter we omit the adjective “complex” for complex vector bundles and complex modules since we work in the complex category. Let  $G$  be a compact Lie group and let  $H$  be a closed normal subgroup of  $G$ . Given a character  $\chi$  of  $H$  and  $g \in G$ , a new character  ${}^g\chi$  of  $H$  is defined by  ${}^g\chi(h) = \chi(g^{-1}hg)$  for  $h \in H$ . This defines an action of  $G$  on the set  $\text{Irr}(H)$  of characters of irreducible  $H$ -modules. Since a character is a class function,  $H$  acts on  $\text{Irr}(H)$  trivially. Therefore, the isotropy subgroup of  $G$  at  $\chi \in \text{Irr}(H)$ , denoted by  $G_\chi$ , contains  $H$ . We choose a representative from each  $G$ -orbit in  $\text{Irr}(H)$  and denote by  $\text{Irr}(H)/G$  the set of those representatives.

Let  $X$  be a connected  $G$ -space on which  $H$  acts trivially. Then all the fibers of a  $G$ -vector bundle  $E$  over  $X$  are isomorphic as  $H$ -modules. We call the unique (up to isomorphism)  $H$ -module the *fiber  $H$ -module* of  $E$ . As is well-known,  $E$  decomposes according to irreducible  $H$ -modules. For  $\chi \in \text{Irr}(H)$ , we denote by  $E(\chi)$  the  $\chi$ -isotypical component of  $E$ , that is, the largest  $H$ -subbundle of  $E$  with a multiple of  $\chi$  as the character of the fiber  $H$ -module. Note that  $gE(\chi)$ , that is  $E(\chi)$  mapped by  $g \in G$ , is  ${}^g\chi$ -isotypical component of  $E$ . This means that  $E(\chi)$  is actually a  $G_\chi$ -vector bundle and that  $\bigoplus_{\lambda \in G(\chi)} E(\lambda)$ , where  $G(\chi)$  denotes the  $G$ -orbit of  $\chi$ , is a  $G$ -subbundle of  $E$ . Since  $\bigoplus_{\lambda \in G(\chi)} E(\lambda)$  is nothing but the induced  $G$ -vector bundle  $\text{ind}_{G_\chi}^G E(\chi)$ , we have the following decomposition

$$E = \bigoplus_{\chi \in \text{Irr}(H)/G} \text{ind}_{G_\chi}^G E(\chi)$$

as  $G$ -vector bundles.

**Lemma 2.1.** *Two  $G$ -vector bundles  $E$  and  $E'$  over  $X$  are isomorphic if and only if  $E(\chi)$  and  $E'(\chi)$  are isomorphic as  $G_\chi$ -vector bundles for each  $\chi \in \text{Irr}(H)/G$ . In particular,  $E$  is trivial if and only if  $E(\chi)$  is trivial for each  $\chi \in \text{Irr}(H)/G$ .*

*Proof.* The necessity is obvious since a  $G$ -vector bundle isomorphism  $E \rightarrow E'$  restricts to a  $G_\chi$ -vector bundle isomorphism  $E(\chi) \rightarrow E'(\chi)$ , and the sufficiency follows from the fact that  $\text{ind}$  is functorial.  $\square$

The observation above can be restated as follows. Denote by  $\text{Vect}_G(X)$  the set of isomorphism classes of  $G$ -vector bundles over  $X$ , and by  $\text{Vect}_{G_\chi}(X, \chi)$  the subset of  $\text{Vect}_{G_\chi}(X)$  with a multiple of  $\chi$  as the character of fiber  $H$ -modules. They are semi-groups under Whitney sum. Then the map sending  $E$  to  $\prod_{\chi \in \text{Irr}(H)/G} E(\chi)$  gives a semi-group isomorphism

$$\Phi: \text{Vect}_G(X) \rightarrow \prod_{\chi \in \text{Irr}(H)/G} \text{Vect}_{G_\chi}(X, \chi).$$

This reduces the study of  $\text{Vect}_G(X)$  to that of  $\text{Vect}_{G_\chi}(X, \chi)$ .

**Lemma 2.2.** *If there is a  $G_\chi$ -vector bundle over  $X$  with  $\chi$  as the character of the fiber  $H$ -module, then  $\text{Vect}_{G_\chi}(X, \chi)$  is isomorphic to  $\text{Vect}_{G_\chi/H}(X)$  as semi-groups. In fact, if  $L$  is such a  $G_\chi$ -vector bundle, then the map*

$$\text{Vect}_{G_\chi}(X, \chi) \rightarrow \text{Vect}_{G_\chi/H}(X)$$

*sending  $E$  to  $\text{Hom}_H(L, E)$  gives an isomorphism.*

*Proof.* It is easy to check that the map

$$\text{Vect}_{G_\chi/H}(X) \rightarrow \text{Vect}_{G_\chi}(X, \chi)$$

sending  $F \in \text{Vect}_{G_\chi/H}(X)$  to  $L \otimes F$  gives the inverse of the map in the lemma.  $\square$

*Remark.* The lemma above does not hold in the real category in general, but it does if  $\chi$  is of real type, i.e., if  $\chi$  is the character of a real irreducible  $H$ -module with the endomorphism algebra isomorphic to  $\mathbb{R}$ .

We conclude this section with the following well-known fact.

**Proposition 2.3.** *If the  $G$ -action on  $X$  is transitive, i.e.,  $X$  is homeomorphic to  $G/K$  for a closed subgroup  $K$ , then any  $G$ -vector bundle over  $X$  is of the form*

$$G \times_K W \rightarrow G/K = X$$

*for some  $K$ -module  $W$ . In fact,  $W$  is the fiber over a point of  $X$  with  $K$  as the isotropy subgroup.*

The proposition above implies that there is an isomorphism

$$\text{Vect}_G(G/K) \cong \text{Vect}_K(*),$$

where  $*$  denotes the one-point  $K$ -space.

### 3. FIBER $H$ -MODULES AND EXTENSION

We say that a character  $\chi$  of  $H$  is  $G$ -invariant if  ${}^g\chi = \chi$  for all  $g \in G$ . The following proposition gives a necessary condition for an  $H$ -module to be the fiber  $H$ -module of a  $G$ -vector bundle  $E$  over a connected  $G$ -space  $X$ .

**Proposition 3.1.** *The character of the fiber  $H$ -module of  $E$  is  $G$ -invariant.*

*Proof.* Let  $x \in X$  and  $g \in G$ . We know that the fibers  $E_x$  and  $E_{gx}$  of  $E$  at  $x$  and  $gx$  are isomorphic as  $H$ -modules. On the other hand, since  $g(g^{-1}hg)v = hgv$  for  $v \in E_x$ , the map  $g: E_x \rightarrow E_{gx}$  becomes an  $H$ -equivariant isomorphism if we consider an  $H$ -action on  $E_x$  given through an automorphism of  $H$  given by  $h \rightarrow g^{-1}hg$ . This implies the proposition.  $\square$

For a group  $K$  containing  $H$ , we say that an  $H$ -module  $V$  *extends* to a  $K$ -module  $W$  (or  $W$  is a  $K$ -extension of  $V$ ) if the restriction  $\text{res}_H W$  of  $W$  to  $H$  is isomorphic to  $V$ .

There are two reasons why we are concerned with the extension of an  $H$ -module. One is that if  $V$  is the fiber  $H$ -module of  $E$  and there is a point in the base space with the isotropy subgroup  $K$  larger than  $H$ , then the fiber over the point gives a  $K$ -extension of  $V$ . The other is that if  $E$  is trivial, i.e., isomorphic to a product bundle with a  $G$ -module as its fiber, then the fiber  $H$ -module of  $E$  must extend to the  $G$ -module.

If  $H$  is a normal subgroup of  $K$  and an  $H$ -module has a  $K$ -extension, then its character must be  $K$ -invariant because a character is a class function. But the converse does not hold in general. The following proposition gives an answer to the converse problem when  $K/H$  is isomorphic to a subgroup of  $O(2)$ .

**Proposition 3.2.** *Let  $H$  be a normal subgroup of  $K$  and let  $V$  be an irreducible  $H$ -module with  $K$ -invariant character.*

- (1) *If  $K/H$  is finite cyclic or isomorphic to  $SO(2)$ , then  $V$  has a  $K$ -extension.*
- (2) *If  $K/H$  is isomorphic to  $O(2)$ , then  $V$  has two  $K$ -extensions or none.*
- (3) *If  $K/H$  is a dihedral group  $D_n$  of order  $2n$ , then*
  - (i) *in case  $n$  is odd,  $V$  has two  $K$ -extensions,*
  - (ii) *in case  $n$  is even,  $V$  has either four  $K$ -extensions or none.*

*In any case, if  $W$  is a  $K$ -extension of  $V$ , then any  $K$ -extension of  $V$  is of the form  $W \otimes U$ , where  $U$  is a one-dimensional  $K/H$ -module viewed as a  $K$ -module through the projection  $K \rightarrow K/H$ , and different  $U$ 's produce different  $K$ -extensions. Therefore, if  $V$  has a  $K$ -extension, then the number of  $K$ -extensions of  $V$  agrees with the number of one-dimensional  $K/H$ -modules.*

*Remark.* (1) The character of an  $H$ -module is  $K$ -invariant whenever  $K/H$  is isomorphic to  $SO(2)$  because  $SO(2)$  is connected. So the  $K$ -invariance of the character of  $V$  in the proposition above is unnecessary in this case, i.e., any  $H$ -module extends to a  $K$ -module whenever  $K/H$  is isomorphic to  $SO(2)$ .

(2) The proposition above does not hold in the real category but it does when  $K/H$  is cyclic and of odd order.

*Proof.* We shall prove the latter statement in the proposition. In this proof, we do not need the assumption that  $K/H$  is isomorphic to a subgroup of  $O(2)$ . Suppose that  $V$  has

$K$ -extensions  $W$  and  $W'$ . Then  $\text{Hom}_H(W, W')$  is a  $K/H$ -module and one-dimensional by Schur's lemma because  $\text{res}_H W = \text{res}_H W' = V$  is irreducible. One can easily check that the map

$$W \otimes \text{Hom}_H(W, W') \rightarrow W'$$

sending  $w \otimes f$  to  $f(w)$  is a  $K$ -linear isomorphism. Therefore, any  $K$ -extension of  $V$  is the tensor product of  $W$  and a one-dimensional  $K/H$ -module viewed as a  $K$ -module. Moreover, since  $\text{Hom}_H(W, W \otimes U) \cong U$  as  $K/H$ -modules, different one-dimensional  $K/H$ -modules produce different  $K$ -extensions of  $V$ . This proves the latter statement.

It is well-known and easy to see that there are two  $O(2)$ - or  $D_n$ -modules of dimension one for  $n$  odd, and four  $D_n$ -modules of dimension one for  $n$  even. This proves the statements on the number of  $K$ -extensions in (2) and (3).

It remains to see that  $V$  has a  $K$ -extension in the cases (1) and (3-i). The case (1) is well-known if  $K/H$  is finite cyclic, see for instance [Isa76]. The book [Isa76] treats only finite groups (so that  $H$  is finite), but some direct proofs work even if  $H$  is infinite. The reader can find one of the direct proofs in [CKS99]. We suspect that the case (1) is known even when  $K/H$  is isomorphic to  $SO(2)$ , but there is no literature as far as we know. Since the proof we found is rather long and has independent interest, we will give it in [CKS99]. The case (3-i) is also well-known, see [Isa76] or [CKS99] for an elementary proof.  $\square$

It turns out that the above facts on representation theory greatly influence the semi-group structure on  $\text{Vect}_G(X)$  when  $X$  is a circle with  $G$ -action.

#### 4. FIBER $H$ -MODULES OF $G$ -VECTOR BUNDLES OVER A CIRCLE

Henceforth, we restrict our concern to  $G$ -vector bundles over a circle with  $G$ -action. For an orthogonal representation  $\rho: G \rightarrow O(2)$  of  $G$ , we denote by  $S(\rho)$  the unit circle of the corresponding representation space. It is well-known that a circle with continuous (resp. smooth)  $G$ -action is equivariantly homeomorphic (resp. diffeomorphic) to  $S(\rho)$  for some representation  $\rho$  [Sch84]. We identify  $S(\rho)$  with the unit circle of the complex plane  $\mathbb{C}$ , and denote a point in  $S(\rho)$  by  $z \in \mathbb{C}$  with absolute value 1. Set  $H = \rho^{-1}(1)$ .

Let us observe the subgroup  $\rho(G)$  of  $O(2)$ . If  $\rho(G)$  is infinite, then it is either  $SO(2)$  or  $O(2)$  itself. Otherwise  $\rho(G)$  is a finite cyclic or dihedral group. Suppose  $\rho(G)$  is a finite cyclic subgroup  $Z_n$  of  $SO(2)$  of order  $n \geq 1$ . Choose and fix an element  $a \in G$  such that  $\rho(a)$  is the rotation through an angle  $2\pi/n$ . Then  $G$  is generated by  $H$  and  $a$  under the relation  $a^n \in H$ , and all the isotropy subgroups  $G_z$  at  $z \in S(\rho)$  are equal to  $H$ . If  $\rho(G)$  is a proper subgroup of  $O(2)$  not contained in  $SO(2)$ , then we may assume that  $\rho(G)$  is a dihedral subgroup  $D_n$  of  $O(2)$  generated by the reflection matrix about the  $x$ -axis and the rotation matrix through an angle  $2\pi/n$ . Choose and fix one more element  $b \in G$  such that  $\rho(b)$  is the reflection matrix about the  $x$ -axis. Then  $G$  is generated by  $H$ ,  $a$  and  $b$  under the relations,  $a^n$ ,  $b^2$ , and  $(ab)^2 \in H$ . The isotropy subgroup  $G_1$  at  $1 \in S(\rho)$  is generated by  $H$  and  $b$ , and  $G_\mu$  at  $\mu = e^{\pi i/n} \in S(\rho)$  is generated by  $H$  and  $ab$ .

Here is a characterization of the fiber  $H$ -modules of  $G$ -vector bundles over  $S(\rho)$ .

**Theorem A.** *An  $H$ -module is the fiber  $H$ -module of a  $G$ -vector bundle over  $S(\rho)$  if and only if its character is  $G$ -invariant.*

*Proof.* The necessity follows from Proposition 3.1, so we prove the sufficiency. Let  $V$  be an  $H$ -module with  $G$ -invariant character. We distinguish three cases according to  $\rho(G)$ .

*Case 1:* The case where  $\rho(G)$  is infinite, i.e.,  $\rho(G) = SO(2)$  or  $O(2)$ . In this case the  $G$ -action on  $S(\rho)$  is transitive, and the isotropy subgroup  $K$  of a point in  $S(\rho)$  is  $H$  when  $\rho(G) = SO(2)$ , and contains  $H$  as an index two subgroup when  $\rho(G) = O(2)$ . Therefore  $V$  has a  $K$ -extension by Proposition 3.2 since the character of  $V$  is  $G$ -invariant (in particular,  $K$ -invariant) and  $K/H$  is trivial or of order two. This together with Proposition 2.3 implies the existence of a  $G$ -vector bundle over  $S(\rho)$  with  $V$  as the fiber  $H$ -module.

*Case 2:* The case where  $\rho(G) = Z_n$ . In this case  $H = G_1 = G_\mu$ . Set  $W = \text{ind}_H^G V$  and consider the product  $G$ -vector bundle  $\underline{W} = S(\rho) \times W$ . Choose an  $H$ -submodule isomorphic to  $V$  in the fiber of  $\underline{W}$  at 1, and identify it with  $V$ . The  $G$ -invariance of the character of  $V$  implies that  $\text{res}_H V \cong \text{res}_H(aV)$ . Viewing  $V$  and  $aV$  as  $H$ -invariant subspaces of  $W$ , one can connect  $V$  and  $aV$  through a continuous family of  $H$ -invariant subspaces along the arc of  $S(\rho)$  joining 1 and  $e^{2\pi i/n}$ , in other words, for each  $z$  in the arc one can find an  $H$ -invariant subspace in the fiber of  $\underline{W}$  at  $z$  so that the family of  $H$ -invariant subspaces varies continuously on the points  $z$ . This is always possible because the set of such  $H$ -invariant subspaces of  $W$  is homeomorphic to a product of Grassmann manifolds which is arcwise connected. Translating the family of  $H$ -invariant subspaces by the action of  $a$  repeatedly yields the desired  $G$ -subbundle of  $\underline{W}$ .

*Case 3:* The case where  $\rho(G) = D_n$ . By Proposition 3.2, there exist  $G_1$ - and  $G_\mu$ -extensions  $V_1$  and  $V_\mu$  of  $V$  respectively. Set  $W = \text{ind}_{G_1}^G V_1 \oplus \text{ind}_{G_\mu}^G V_\mu$  and consider the product  $G$ -vector bundle  $\underline{W}$ . Then  $V_1$  and  $V_\mu$  are contained as  $G_1$ - and  $G_\mu$ -submodules in the fibers of  $\underline{W}$  at 1 and  $\mu$ , respectively. Since  $\text{res}_H V_1 \cong \text{res}_H V_\mu$ , it is possible to connect  $V_1$  and  $V_\mu$  through a continuous family of  $H$ -invariant subspaces along the arc of  $S(\rho)$  joining 1 and  $\mu$  as we did in Case 2 above. We translate it using the action of  $b$  and then using the action of  $a$  repeatedly to obtain the desired  $G$ -subbundle of  $\underline{W}$ .  $\square$

*Remark.* (1) The proof above shows that any  $G_1$ -extension (and  $G_\mu$ -extension when  $\rho(G) = D_n$ ) of  $V$  can be realized as the fiber at 1 (and at  $\mu$  when  $\rho(G) = D_n$ ) of a  $G$ -vector bundle over  $S(\rho)$ .

(2) The proof above also works in the real category except the extension problem of  $V$ . Namely, since Proposition 3.2 does not hold in the real category, we need to assume that, in addition to the  $G$ -invariance of the character of  $V$ ,  $V$  has a  $G_1$ -extension when  $\rho(G) = O(2)$ , and both  $G_1$ - and  $G_\mu$ -extensions when  $\rho(G) = D_n$ .

## 5. THE SEMI-GROUP STRUCTURE ON $\text{Vect}_G(S(\rho))$

In this section we determine the semi-group structure on  $\text{Vect}_G(S(\rho))$ . We begin with a simple case.

**Lemma 5.1.** *Suppose the  $G$ -action on  $S(\rho)$  is effective, in other words,  $\rho: G \rightarrow O(2)$  is injective. Then the semi-group  $\text{Vect}_G(S(\rho))$  is generated by*

- (1) *one trivial  $G$ -line bundle if  $\rho(G) \subset SO(2)$ ,*
- (2) *two trivial  $G$ -line bundles  $L^\pm$  if  $\rho(G) = O(2)$ , and*
- (3) *four  $G$ -line bundles  $L^{\pm\pm}$  with a relation  $L^{++} + L^{--} = L^{+-} + L^{-+}$  otherwise.*

*Proof.* Since  $\rho$  is injective,  $P = \rho^{-1}(SO(2))$  acts freely on  $S(\rho)$ ; so taking orbit spaces by  $P$  gives an isomorphism

$$\mathrm{Vect}_G(S(\rho)) \cong \mathrm{Vect}_{G/P}(S(\rho)/P).$$

In fact, the inverse is given by pulling back elements in  $\mathrm{Vect}_{G/P}(S(\rho)/P)$  by the quotient map from  $S(\rho)$  to  $S(\rho)/P$ . Because of this isomorphism, it suffices to study the semi-group structure on  $\mathrm{Vect}_{G/P}(S(\rho)/P)$ . Note that  $S(\rho)/P$  is again a circle or a point, and that the pullback of a trivial bundle is again trivial.

(1) The case where  $\rho(G) \subset SO(2)$ . In this case  $P = G$ , i.e.,  $G/P$  is the trivial group, so the semi-group  $\mathrm{Vect}_{G/P}(S(\rho)/P)$  is generated by one element, that is the trivial line bundle, as is well known. This implies (1) in the lemma.

(2) The case where  $\rho(G) = O(2)$ . In this case  $G/P$  is of order two and  $S(\rho)/P$  is a point. Therefore,  $\mathrm{Vect}_{G/P}(S(\rho)/P)$  is generated by two elements of dimension one. This implies (2) in the lemma.

(3) The case where  $\rho(G) = D_n$  for some  $n$ . In this case  $S(\rho)/P$  is again a circle,  $G/P$  is of order two, and the action of  $G/P$  on  $S(\rho)/P$  is a reflection. In the sequel it suffices to treat the case where  $n = 1$ . But this case is already studied in [Kim94]. (Kim treats real bundles but the same argument works for complex bundles.) The result in [Kim94] says that  $D_1$ -vector bundles over  $S(\rho)$  are distinguished by the fiber  $D_1$ -modules over the fixed points  $\pm 1 \in S(\rho)$ , and that any pair of  $D_1$ -modules of the same dimension is realized as the fiber  $D_1$ -modules at  $\pm 1$  of a  $D_1$ -vector bundle over  $S(\rho)$ . Since  $D_1$  is of order two, there are two one-dimensional  $D_1$ -modules (one is the trivial one  $\mathbb{C}_+$  and the other is the nontrivial one  $\mathbb{C}_-$ ) and that any  $D_1$ -module is a direct sum of them. Therefore, there are four inequivalent  $D_1$ -line bundles  $L^{\pm\pm}$ , where  $L^{\epsilon\delta}$  ( $\epsilon$  and  $\delta$  stand for  $+$  or  $-$ ) denotes the  $D_1$ -line bundle with the fiber  $D_1$ -modules  $\mathbb{C}_\epsilon$  at 1 and  $\mathbb{C}_\delta$  at  $-1$ , and the structure of  $\mathrm{Vect}_{D_1}(S(\rho))$  is as stated in (3).

For the reader's convenience, we shall give the argument in [Kim94] briefly. First we observe that any pair of one-dimensional  $D_1$ -modules can be realized as the fiber  $D_1$ -modules at  $\pm 1$  of a  $D_1$ -line bundle over  $S(\rho)$ . In fact, the trivial (real) line bundle and the (real) Hopf line bundle have respectively two different  $D_1$ -liftings to the total space, and each  $D_1$ -lifting of the trivial (real) line bundle has the same fiber  $D_1$ -modules at  $\pm 1$  while that of the (real) Hopf line bundle has different fiber  $D_1$ -modules at  $\pm 1$ . We consider complexification of them. Then any pair of  $D_1$ -modules of dimension  $m$  can be realized as the fiber  $D_1$ -modules at  $\pm 1$  by taking the Whitney sum of suitable  $m$  number of those complexified  $D_1$ -line bundles. On the other hand, the same technique used in the proof of Theorem A shows that any  $D_1$ -vector bundle over  $S(\rho)$  decomposes into the Whitney sum of the above  $D_1$ -line bundles.  $\square$

Theorem A applied with  $G = G_\chi$  and the irreducible  $H$ -module with character  $\chi$  says that the assumption in Lemma 2.2 is satisfied when  $X = S(\rho)$ , so  $\mathrm{Vect}_{G_\chi}(S(\rho), \chi)$  has the same semi-group structure as  $\mathrm{Vect}_{G_\chi/H}(S(\rho))$ . Here the action of  $G_\chi/H$  on  $S(\rho)$  is effective, so the lemma above can be applied to  $\mathrm{Vect}_{G_\chi/H}(S(\rho))$ . In the sequel



the semi-group structure on  $\text{Vect}_{G_\chi}(S(\rho), \chi)$  is divided into three types depending on  $\rho(G_\chi)$ . We denote the generators of  $\text{Vect}_{G_\chi}(S(\rho), \chi)$  corresponding to the generators in Lemma 5.1 by

$$\begin{cases} L_\chi, & \text{if } \rho(G_\chi) \subset SO(2), \\ L_\chi^\pm, & \text{if } \rho(G_\chi) = O(2), \\ L_\chi^{\pm\pm}, & \text{otherwise.} \end{cases}$$

The following theorem follows immediately from Lemma 5.1.

**Theorem B.** *The semi-group  $\text{Vect}_{G_\chi}(S(\rho), \chi)$  is generated by*

- (1) *one element  $L_\chi$  if  $\rho(G_\chi) \subset SO(2)$ ,*
- (2) *two elements  $L_\chi^\pm$  if  $\rho(G_\chi) = O(2)$ , and*
- (3) *four elements  $L_\chi^{\pm\pm}$  with the relation  $L_\chi^{++} + L_\chi^{--} = L_\chi^{+-} + L_\chi^{-+}$  otherwise.*  $\square$

*Remark.* If  $\rho(G) \subset SO(2)$ , then  $\rho(G_\chi)$  is of type (1) above for any  $\chi$ . If  $\rho(G) = O(2)$ , then  $\rho(G_\chi)$  is of type (1) or (2) above; more precisely  $\rho(G_\chi) = SO(2)$  or  $O(2)$  because the  $G$ -action on  $\text{Irr}(H)$  reduces to an action of  $G/H = \rho(G) = O(2)$  and the action of  $SO(2)$  on  $\text{Irr}(H)$  is trivial since  $SO(2)$  is connected. Moreover, if  $\rho(G) = D_n$ , then  $\rho(G_\chi)$  is of type (1) or (3) above. Therefore, the semi-group structure on  $\text{Vect}_G(S(\rho))$  can be read from the theorem above and the isomorphism  $\Phi$  in Section 2.

Using the theorem above, one can easily enumerate  $G$ -vector bundles over  $S(\rho)$  with a fixed  $H$ -module  $V$  as the fiber  $H$ -module. Since  $V$  must have a  $G$ -invariant character by Theorem A, one can express the character of  $V$  as

$$\sum_{\chi \in \text{Irr}(H)/G} m_\chi \left( \sum_{\lambda \in G(\chi)} \lambda \right)$$

with non-negative integers  $m_\chi$ , where  $m_\chi$ 's are zero for all but finitely many  $\chi$ 's in  $\text{Irr}(H)/G$  because  $V$  is of finite dimension. Set

$$e(\chi) = \begin{cases} 0, & \text{if } \rho(G_\chi) \subset SO(2) \\ 1, & \text{if } \rho(G_\chi) = O(2) \\ 2, & \text{otherwise.} \end{cases}$$

With this understood

**Corollary 5.2.** *The number of isomorphism classes of  $G$ -vector bundles over  $S(\rho)$  with  $V$  as the fiber  $H$ -modules is given by  $\prod_{\chi \in \text{Irr}(H)/G} (m_\chi + 1)^{e(\chi)}$ .*  $\square$

## 6. ISOMORPHISM THEOREM

In this section we present another approach to study the semi-group structure on  $\text{Vect}_G(S(\rho))$ . The following theorem, which we call an isomorphism theorem, reduces the study of  $\text{Vect}_G(S(\rho))$  to representation theory.

**Theorem 6.1.** *Two  $G$ -vector bundles  $E$  and  $E'$  over  $S(\rho)$  are isomorphic if and only if the fiber  $G_z$ -modules  $E_z$  and  $E'_z$  at  $z \in S(\rho)$  are isomorphic for  $z = 1$  (and for  $z = \mu$  when  $\rho(G) = D_n$ ).*

*Proof.* The necessity part is obvious, so we prove the sufficiency. We note that if there exists an equivariant isomorphism  $\Psi: E \rightarrow E'$ , then it must satisfy the equivariance condition

$$\Psi_{\rho(g)z} = g\Psi_z g^{-1}$$

for any  $g \in G$  where  $\Psi_z = \Psi|_{E_z}$ . By the assumption we have a  $G_1$ -linear isomorphism  $\Psi_1$  (and a  $G_\mu$ -linear isomorphism  $\Psi_\mu$  when  $\rho(G) = D_n$ ). In the following we will define  $\Psi_z$  for all  $z \in S(\rho)$  using the above equivariance condition to get an equivariant isomorphism  $\Psi$ . We consider three cases according to the images of  $G$  by  $\rho$ .

*Case 1:* The case where  $\rho(G) = SO(2)$  or  $O(2)$ . In this case the  $G$ -action on  $S(\rho)$  is transitive, so for any  $z \in S(\rho)$  we define  $\Psi_z = g\Psi_1 g^{-1}$  with  $g \in G$  such that  $z = \rho(g)1$ . The well-definedness follows from the  $G_1$ -equivariance of  $\Psi_1$ . This gives the desired equivariant isomorphism  $\Psi$ .

*Case 2:* The case where  $\rho(G) = Z_n$ . Let  $\zeta = e^{2\pi i/n}$  and define  $\Psi_\zeta$  by  $a\Psi_1 a^{-1}$ . The map  $\Psi_\zeta$  is also an  $H$ -equivariant isomorphism. We connect  $\Psi_1$  and  $\Psi_\zeta$  along the arc of  $S(\rho)$  joining 1 and  $\zeta$ , in other words, we find an  $H$ -equivariant linear isomorphism  $\Psi_z$  for each  $z$  in the arc of  $S(\rho)$  so that  $\Psi_z$  is continuous at those  $z$ . (This is always possible because the set of  $H$ -linear isomorphisms between  $E_z$  and  $E'_z$  is arcwise connected, in fact, homeomorphic to a product of  $GL(N, \mathbb{C})$ 's.) Now we define  $\Psi_z$  for any  $z \in S(\rho)$  using the equivariance condition  $\Psi_{\zeta z} = a\Psi_z a^{-1}$ . This gives the desired isomorphism  $\Psi$ .

*Case 3:* The case where  $\rho(G) = D_n$ . Note that the equivariance condition of  $\Psi$  is

$$\Psi_z = h\Psi_z h^{-1} \text{ for any } h \in H, \quad \Psi_{\zeta z} = a\Psi_z a^{-1}, \quad \Psi_{\bar{z}} = b\Psi_z b^{-1}.$$

We connect  $\Psi_1$  and  $\Psi_\mu$  along the arc joining 1 and  $\mu$  to obtain  $\Psi_z$  for  $z$  in the arc. Then using the equivariance condition  $\Psi_{\bar{z}} = b\Psi_z b^{-1}$ , we define  $\Psi_z$  for  $z$  in the arc joining 1 and  $\mu^{-1}$ . Thus we have defined  $\Psi_z$  for  $z$  in the arc joining  $\mu^{-1}$  and  $\mu$ . We then define  $\Psi_z$  for all  $z$  using the equivariance condition  $\Psi_{\zeta z} = a\Psi_z a^{-1}$ . This gives the desired isomorphism.  $\square$

For a group  $K$  we denote by  $\text{Rep}(K)$  the set of isomorphism classes of  $K$ -modules, and by  $\text{Rep}(G_1, G_\mu)$  the set of elements  $(V, W) \in \text{Rep}(G_1) \times \text{Rep}(G_\mu)$  with  $\text{res}_H V = \text{res}_H W$ . Restriction of a  $G$ -vector bundle over  $S(\rho)$  to fibers at 1 (and  $\mu$  when  $\rho(G) = D_n$ ) yields a map

$$\Gamma: \text{Vect}_G(S(\rho)) \rightarrow \begin{cases} \text{Rep}(H)^G & \text{if } \rho(G) \subset SO(2), \\ \text{Rep}(G_1) & \text{if } \rho(G) = O(2), \\ \text{Rep}(G_1, G_\mu) & \text{if } \rho(G) = D_n \end{cases}$$

where  $\text{Rep}(H)^G$  denotes the subset of  $\text{Rep}(H)$  with  $G$ -invariant character. The target of the map  $\Gamma$  is a semi-group under direct sum. With this understood

**Proposition 6.2.** *The map  $\Gamma$  is an isomorphism.*

*Proof.* It is obvious that  $\Gamma$  is a homomorphism and that the characters of all representations in  $\text{Rep}(G_1)$  (when  $\rho(G) = O(2)$ ) and  $\text{Rep}(G_1, G_\mu)$  (when  $\rho(G) = D_n$ ) are  $G$ -invariant. The surjectivity follows from Theorem A (when  $\rho(G) \subset SO(2)$ ) and the

first remark following it in Section 4 (when  $\rho(G) = O(2)$  or  $D_n$ ), and the injectivity follows from Theorem 6.1.  $\square$

As a matter of fact, the source and target of the map  $\Gamma$  have more structures, that is, they have products given by tensor product and  $R(G)$  acts on them naturally through the tensor product. In fact,  $R(G)$  acts on the target through the restriction to  $H$ ,  $G_1$ , or  $G_\mu$ . Clearly the map  $\Gamma$  preserves these structures.

## 7. TRIVIALITY OF $G$ -VECTOR BUNDLES OVER A CIRCLE

In this section we investigate when a  $G$ -vector bundle over  $S(\rho)$  is trivial. Here is the criterion of triviality of a  $G$ -vector bundle over  $S(\rho)$ .

**Lemma 7.1.** (1) *A  $G$ -vector bundle over  $S(\rho)$  is trivial if and only if the fiber  $G_z$ -module at  $z = 1$  (and at  $z = \mu$  when  $\rho(G) = D_n$ ) extends to a (same when  $\rho(G) = D_n$ )  $G$ -module.*

(2) *Unless  $\rho(G) \subset SO(2)$ , the number of the isomorphism classes of trivial  $G$ -vector bundles over  $S(\rho)$  with an irreducible fiber  $H$ -module  $V$  agrees with the number of  $G$ -extensions of  $V$ .*

*Proof.* (1) The necessity is trivial and the sufficiency follows from Theorem 6.1.

(2) Let  $W$  and  $W'$  be two  $G$ -extensions of  $V$  and suppose that the product bundles  $\underline{W}$  and  $\underline{W}'$  are isomorphic. Then  $\text{res}_{G_1} W \cong \text{res}_{G_1} W'$  (and  $\text{res}_{G_\mu} W \cong \text{res}_{G_\mu} W'$  if  $\rho(G) = D_n$ ). It is easy to see from Proposition 3.2 that each  $G$ -extension of  $V$  is distinguished by its restriction to  $G_1$  (and  $G_\mu$  if  $\rho(G) = D_n$ ). Therefore,  $W$  and  $W'$  are isomorphic as  $G$ -modules, proving (2).  $\square$

**Theorem C.** *The triviality of the generators appeared in Theorem B is as follows.*

- (1)  $L_\chi$  is trivial.
- (2)  $L_\chi^\pm$  are both trivial or both nontrivial.
- (3) Two of  $L_\chi^{\pm\pm}$  are trivial and the other two are nontrivial if  $|\rho(G_\chi)|/2$  is odd, and  $L_\chi^{\pm\pm}$  are all trivial or all nontrivial if  $|\rho(G_\chi)|/2$  is even.

*Proof.* This follows from Proposition 3.2 and Lemma 7.1.  $\square$

In fact,  $L_\chi^\pm$  are related through the tensor product with a one-dimensional nontrivial representation  $G_\chi \rightarrow O(2)/SO(2) = \{\pm 1\}$  (which is  $\rho$  composed with the projection  $O(2) \rightarrow O(2)/SO(2)$ ), and  $L_\chi^{\pm\pm}$  are related through the tensor product with four  $G_\chi$ -line bundles  $L^{\pm\pm}$  which are pullback of the four  $D_1$ -line bundles by the quotient map from  $S(\rho)$  to  $S(\rho)/P$  where  $P = \rho^{-1}(SO(2))$  as before. Note that the fiber  $H$ -modules of  $L_\chi^{\pm\pm}$  are trivial. The  $G_\chi$ -line bundles  $L^{\pm\pm}$  are well understood and one (actually two) of  $L_\chi^{\pm\pm}$  is trivial in case  $n$  is odd, so we completely understand  $L_\chi^{\pm\pm}$  in this case. But we do not know  $L_\chi^{\pm\pm}$  explicitly when  $n$  is even and they are all nontrivial.

**Corollary 7.2.** *If  $\rho(G_\chi) \subset SO(2)$ , then every element in  $\text{Vect}_{G_\chi}(S(\rho), \chi)$  is trivial.*

*Proof.* The corollary follows from Theorem B (1) and Theorem C (1).  $\square$

**Corollary D.** *Every  $G$ -vector bundle over  $S(\rho)$  is trivial if  $\rho(G) \subset SO(2)$ .*

*Proof.* If  $\rho(G) \subset SO(2)$ , then  $\rho(G_\chi) \subset SO(2)$  for any  $\chi$ . Therefore the corollary follows from Lemma 2.1 and Corollary 7.2.  $\square$

## 8. DESCRIPTION OF $G$ -LINE BUNDLES OVER A CIRCLE

In this section, we describe  $L_\chi^{\pm\pm}$  explicitly when  $\chi$  is the character of a one-dimensional  $H$ -module. In the following,  $\varphi$  denotes a one-dimensional  $H$ -representation with  $G$ -invariant character  $\chi$ . (Actually  $\chi$  agrees with  $\varphi$  since  $\varphi$  is one-dimensional.) To simplify notation we denote  $G_\chi$  by  $G$ .

When  $\rho(G) = D_n$ , let  $a$  and  $b$  be as before, i.e., they denote elements of  $G$  whose images by  $\rho$  are respectively the rotation through an angle  $2\pi/n$  and the reflection by  $x$ -axis. Note that  $G$  is generated by  $H$ ,  $a$  and  $b$  under the relations  $a^n$ ,  $b^2$ , and  $(ab)^2 \in H$ .

**Lemma 8.1.** *Suppose  $\rho(G) = D_n$ . Then*

- (1)  $\varphi(a^n)^2 = \varphi(abab^{-1})^n$ ,
- (2) *when  $n$  is even,  $\varphi$  has a  $G$ -extension if and only if  $\varphi(a^n) = \varphi(abab^{-1})^{n/2}$ .*

*Remark.* When  $n$  is odd, we know that  $\varphi$  has a  $G$ -extension by Proposition 3.2 (3). It can also be seen from the proof of (2) below.

*Proof.* Let  $\tilde{\varphi}: G_1 \rightarrow \text{GL}(1, \mathbb{C})$  be an extension of  $\varphi$  to  $G_1$ . Since (the character of)  $\varphi$  is  $G$ -invariant (in particular,  $G_1$ -invariant) and  $H$  is an index two subgroup of  $G_1$ , such an extension exists by Proposition 3.2 (1).

(1) It is elementary to see that  $a^i b a^i \in G_1$  and  $a^i b a^i a b a \in H$  for  $1 \leq i \leq n-1$ . Since  $\varphi$  is  $G$ -invariant we have

$$\tilde{\varphi}(a^i b a^i) \tilde{\varphi}(a b a) = \varphi(a^i b a^i a b a) = \varphi(a(a^i b a^i a b a)a^{-1}) = \tilde{\varphi}(a^{i+1} b a^{i+1}) \tilde{\varphi}(b)$$

for  $1 \leq i \leq n-1$ . By an inductive application of the above identity, we have

$$\tilde{\varphi}(a b a)^n = \tilde{\varphi}(a^n b a^n) \tilde{\varphi}(b)^{n-1}.$$

It follows that

$$\varphi(abab^{-1})^n = \tilde{\varphi}(a b a)^n \tilde{\varphi}(b^{-1})^n = \tilde{\varphi}(a^n b a^n) \tilde{\varphi}(b)^{n-1} \tilde{\varphi}(b^{-1})^n = \varphi(a^n)^2.$$

(2) The necessity is obvious, so we shall prove the sufficiency. Let  $A$  be an  $n$ -th root of  $\varphi(a^n)$ . Then  $(A^{-2}\varphi(abab^{-1}))^n = 1$  by the identity in (1) above, so there is an integer  $k$  (determined module  $n$ ) such that  $A^{-2}\varphi(abab^{-1}) = \zeta^k$  where  $\zeta = e^{2\pi i/n}$ . The equality  $\varphi(a^n) = \varphi(abab^{-1})^{n/2}$  is equivalent to  $k$  being even. We define  $\tilde{\varphi}(a) = A\zeta^{k/2}$ . Then  $\tilde{\varphi}(a)^n = (A\zeta^{k/2})^n = A^n = \varphi(a^n)$ . Therefore, to see that the extended  $\tilde{\varphi}$  is a  $G$ -extension of  $\varphi$ , it only remains to check that  $\tilde{\varphi}(a)^2 \tilde{\varphi}(b)^2 = \varphi((ab)^2)$ . (Remember that  $(ab)^2 \in H$ .) The left hand side at the identity is equal to  $A^2 \zeta^k \tilde{\varphi}(b)^2$  while the right hand side is equal to  $\varphi(abab^{-1})\varphi(b^2)$  which agrees with  $A^2 \zeta^k \tilde{\varphi}(b)^2$  because  $A^{-2}\varphi(abab^{-1}) = \zeta^k$  by the choice of  $k$  above.  $\square$

Let  $\varphi: H \rightarrow U(1)$  be a unitary representation with  $G$ -invariant character. If  $\rho(G) = D_n$ , then there are exactly four  $G$ -line bundles with  $\varphi$  as the fiber  $H$ -representation by Theorem B (3). They are described explicitly in the following example.

**Example 8.2.** Assume that  $\rho(G) = D_n$ . Let  $\varphi: H \rightarrow U(1)$  be a unitary representation with  $G$ -invariant character, and let  $\tilde{\varphi}: G_1 \rightarrow U(1)$  be a  $G_1$ -extension of  $\varphi$ . Let  $A$  be an  $n$ -th root of  $\varphi(a^n)$ . As observed in the proof of Lemma 8.1 (2), there is an integer  $k$  such that  $A^{-2}\tilde{\varphi}(abab^{-1}) = \zeta^k$ . One can check that

$$h(z, v) = (z, \varphi(h)v) \text{ for } h \in H, \quad a(z, v) = (\zeta z, Av), \quad \text{and} \quad b(z, v) = (\bar{z}, \tilde{\varphi}(b)z^k v)$$

define an action of  $G$  on  $S^1 \times \mathbb{C}$ . In fact, it defines a  $G$ -line bundle over  $S(\rho)$  such that the fiber representation at 1 is  $\tilde{\varphi}$  and that at  $\mu$  is given by

$$h \mapsto \varphi(h), \quad ab \mapsto A\mu^k \tilde{\varphi}(b).$$

Since the integer  $k$  is only determined modulo  $n$  (once  $A$  is chosen) and  $\mu^n = -1$ , this construction gives two  $G$ -line bundles with  $\tilde{\varphi}$  as the fiber representation at 1. (If we take  $k+n$  instead of  $k$ , then the fiber representation at  $\mu$  evaluated on  $ab$  changes the sign.) Since there are two  $G_1$ -extensions of  $\varphi$  by Proposition 3.2, the above construction describes all the four  $G$ -line bundles over  $S(\rho)$  with  $\varphi$  as the fiber  $H$ -module.

We now have the classification result for  $G$ -line bundles over  $S(\rho)$ .

**Theorem 8.3.** *Let  $\varphi: H \rightarrow U(1)$  be an  $H$ -representation with  $G$ -invariant character. Let  $N$  be the number of  $G$ -line bundles over  $S(\rho)$  with  $\varphi$  as the fiber  $H$ -module.*

- (1) *If  $\rho(G) \subset SO(2)$ , then  $N = 1$  and the bundle is trivial.*
- (2) *If  $\rho(G) = O(2)$ , then  $N = 2$  and both bundles are trivial or both are nontrivial.*
- (3) *If  $\rho(G) = D_n$ , then  $N = 4$  and all the four bundles are given in Example 8.2, and*
  - (i) *if  $n$  is odd, then two of them are trivial and the other two are nontrivial;*
  - (ii) *if  $n$  is even, then all the four bundles are trivial if  $\varphi(a^n) = \varphi(abab^{-1})^{\frac{n}{2}}$ , and all are nontrivial otherwise.*

*Proof.* This follows from Theorem B, Theorem C, and the observation done in Example 8.2.  $\square$

## 9. THE CASE WHEN $G$ IS ABELIAN

When  $G$  is abelian (and hence so is the subgroup  $H$ ), any irreducible  $H$ -module is one-dimensional and  $G_\chi = G$  for any character  $\chi$  of  $H$ . Therefore  $\text{Vect}_G(S(\rho))$  is generated by  $G$ -line bundles. Moreover, since  $\rho(G)$  is an abelian subgroup of  $O(2)$ , it is contained in  $SO(2)$  or isomorphic to  $D_1$  or  $D_2$ . When  $\rho(G) \subset SO(2)$ , any  $G$ -line bundle over  $S(\rho)$  is trivial as we know. When  $\rho(G) = D_2$ , the condition  $\varphi(a^n) = \varphi(abab^{-1})^{n/2}$  in Theorem 8.3 (3-ii) for  $n = 2$  holds because  $G$  is abelian; so any  $G$ -line bundle over  $S(\rho)$  is trivial in this case, too. But there are two nontrivial  $G$ -line bundles when  $\rho(G) = D_1$  as claimed in Theorem 8.3 (3-i). The following example is simply an interpretation of Example 8.2 to the special case when  $\rho(G) = D_1$ .

**Example 9.1.** Suppose  $G$  is abelian and  $\rho(G) = D_1$ . Then  $G = G_1$ . Let  $\varphi: H \rightarrow U(1)$  be a unitary representation of  $H$ . (Any  $\varphi$  is  $G$ -invariant because  $G$  is abelian.) As in Example 8.2 choosing a  $G$ -extension  $\tilde{\varphi}$  of  $\varphi$  induces a  $G$ -action on  $S(\rho) \times \mathbb{C}$  defined by

$$h(z, v) = (z, \varphi(h)v) \text{ for } h \in H \quad \text{and} \quad b(z, v) = (\bar{z}, \tilde{\varphi}(b)zv).$$

It gives a nontrivial  $G$ -line bundle over  $S(\rho)$ . Since there are two  $G$ -extensions of  $\varphi$ , this produces two nontrivial  $G$ -line bundles over  $S(\rho)$  with  $\varphi$  as the fiber  $H$ -module.

Summing up, we have

**Proposition 9.2.** *Suppose  $G$  is abelian and let  $E$  be a  $G$ -vector bundle over  $S(\rho)$ .*

- (1) *If  $\rho(G) \neq D_1$ , then  $E$  is trivial.*
- (2) *If  $\rho(G) = D_1$ , then  $E$  is the Whitney sum of trivial bundles and the nontrivial line bundles in Example 9.1.*

## 10. EQUIVARIANT $K$ -GROUPS OF A CIRCLE

In this section we apply the results discussed in the previous sections to the calculation of the reduced equivariant  $K$ -group of  $S(\rho)$ . For a compact  $G$ -space  $X$ , the equivariant  $K$ -group  $K_G(X)$  of  $X$  is defined to be the Grothendieck group of finite dimensional  $G$ -vector bundles over  $X$ . If  $X$  has a base point  $*$  fixed by the  $G$ -action, then the reduced equivariant  $K$ -group  $\tilde{K}_G(X)$  is defined to be the kernel of the restriction homomorphism  $K_G(X) \rightarrow K_G(*)$  induced from the inclusion map. In fact,  $K_G(X)$  and  $\tilde{K}_G(X)$  are algebras over  $R(G)$  (although there is no identity element in  $\tilde{K}_G(X)$ ).

The additive structure on  $K_G(S(\rho))$  can be determined completely by Lemma 2.1 and Theorem B. One can also describe the  $R(G)$ -algebra structure in terms of representation ring through the map  $\Gamma$  in Section 6. In the following, we shall compute  $\tilde{K}_G(S(\rho))$ . Note that  $\tilde{K}_G(S(\rho))$  is defined only when  $S(\rho)$  has a fixed point, i.e.,  $G = H$  or  $\rho(G) = D_1$ , and that  $\tilde{K}_G(S(\rho))$  is trivial if  $G = H$ . Suppose  $\rho(G) = D_1$ . Then the  $G$ -fixed point set  $S(\rho)^G$  consists of two points  $\{\pm 1\}$  and we take  $-1$  to be a base point. It follows from Theorem 6.1 that the restriction homomorphism

$$\tilde{K}_G(S(\rho)) \rightarrow \tilde{K}_G(S(\rho)^G) \cong R(G)$$

to fibers at 1 is injective. The following theorem determines the image of the homomorphism as an ideal of  $R(G)$ , which extends Y. Yang's result for  $G$  finite cyclic [Yan95, Theorem A] to any compact Lie group  $G$ . Denote by  $\mathbb{C}_+$  and  $\mathbb{C}_-$  the  $G$ -modules of dimension one induced from the trivial and the nontrivial  $D_1$ -modules of dimension one, respectively, by the homomorphism  $G \rightarrow G/H \cong D_1$ .

**Theorem 10.1.** *If  $\rho(G) = D_1$ , then  $\tilde{K}_G(S(\rho))$  is isomorphic to the ideal  $R(G)(\mathbb{C}_+ - \mathbb{C}_-)$  in  $R(G)$  generated by  $\mathbb{C}_+ - \mathbb{C}_-$ . In particular,  $\tilde{K}_G(S(\rho))$  is torsion-free for any compact Lie group  $G$ .*

*Proof.* The remark (1) at the end of Section 4 implies that  $R(G)(\mathbb{C}_+ - \mathbb{C}_-)$  is contained in the image of  $\tilde{K}_G(S(\rho)) \rightarrow \tilde{K}_G(S(\rho)^G)$ , so we prove the converse.

Choose an element  $E - F$  in  $\tilde{K}_G(S(\rho))$ . Then the fibers of  $E$  and  $F$  at the base point  $-1$  are isomorphic as  $G$ -modules. In particular,  $E$  and  $F$  have the same fiber  $H$ -module and thus  $\text{res}_H E_1$  is isomorphic to  $\text{res}_H F_1$ . Hence, one can express the image of  $E - F$ , i.e.,  $E_1 - F_1$  in  $R(G)$  as  $E_1 - F_1 \cong \bigoplus (E_i - F_i)$  where  $E_i$  and  $F_i$  are irreducible  $G$ -submodules of  $E_1$  and  $F_1$ , respectively, such that  $\text{res}_H E_i \cong \text{res}_H F_i$ . Here we note that an irreducible  $G$ -module  $W$  is uniquely determined by  $\text{res}_H W$  if it is reducible, because

$2W \cong \operatorname{ind}_H^G \operatorname{res}_H W$  in this case, see [BtD85, Theorem 7.3 (ii), Chapter VI]. Therefore, if  $\operatorname{res}_H E_i \cong \operatorname{res}_H F_i$  is reducible, then  $E_i - F_i = 0$  in  $R(G)$ . On the other hand, if  $\operatorname{res}_H E_i \cong \operatorname{res}_H F_i$  is irreducible, then both  $E_i$  and  $F_i$  are  $G$ -extensions of  $\operatorname{res}_H E_i$ . Thus  $F_i$  is isomorphic to  $E_i$  or  $E_i \otimes \mathbb{C}_-$  by the last statement of Proposition 3.2. It follows that  $E_i - F_i$  is either zero or  $E_i \otimes (\mathbb{C}_+ - \mathbb{C}_-)$  in  $R(G)$ . Therefore, the image of  $E - F$  is contained in the ideal  $R(G)(\mathbb{C}_+ - \mathbb{C}_-)$ .  $\square$

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DEPARTMENT OF MATHEMATICS, OSAKA CITY UNIVERSITY  
*E-mail address:* chofchof@free.kaist.ac.kr

DEPARTMENT OF APPLIED MATHEMATICS, PAICHAJ UNIVERSITY  
*E-mail address:* sskim@www.paichai.ac.kr

DEPARTMENT OF MATHEMATICS, OSAKA CITY UNIVERSITY  
*E-mail address:* masuda@sci.osaka-cu.ac.jp

DEPARTMENT OF MATHEMATICS, KOREA ADVANCED INSTITUTE OF SCIENCE AND TECHNOLOGY  
*E-mail address:* dysuh@action.kaist.ac.kr